Limite
Rešitve

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1. Računanje limit

Naloga 1.1 Vrednosti se bližajo 1.

x	$f(x)$
1	0,8415...
0,1	0,9983...
0,01	0,999983...
0,001	0,99999983...
0,0001	0,9999999983...

Iz tega lahko sklepamo le, da

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

Ker ne poznamo limite z leve, torej ne moremo govoriti o limiti sami. Toda ker je funkcija soda, $f(-x) = \sin(-x)/(-x) = \sin(x)/x = f(x)$, sta limiti z obeh strani enaki, torej velja

$$\lim_{x\to 0^+} \frac{\sin x}{x} = \lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Naloga 1.2

(a) $\lim_{x\to 2}[x^3 - 2x^2 + 1] = 1$

(b) $\lim_{t\to -2}\left[\frac{t+2}{t^2-4}\right] = \lim_{t\to -2}\left[\frac{(t+2)}{(t+2)(t-2)}\right] = \frac{1}{-2-2} = -\frac{1}{4}$

(c) $\lim_{x\to 1}\left[\frac{x-1}{x^2+x-2}\right] = \lim_{x\to 1}\left[\frac{(x-1)}{(x-1)(x+2)}\right] = \lim_{x\to 1}\left[\frac{1}{x+2}\right] = \frac{1}{3}$

(d) $\lim_{h\to 0}\left[\frac{(6+h)^2-36}{h}\right] = \lim_{h\to 0}\left[\frac{12h+h^2}{h}\right] = \lim_{h\to 0}[12+h] = 12$

(e) $\lim_{z\to 3}\left[\frac{z^2+3z+2}{z^3-5z^2+3z+9}\right] = \lim_{z\to 3}\left[\frac{(z+2)(z+1)}{(z-3)^2(z+1)}\right] = \lim_{z\to 3}\left[\frac{(z+2)}{(z-3)^2}\right] = +\infty$
Limita obstaja le, ker je $z = 3$ pol sodega reda (tam funkcija ne spremeni predznaka, kot recimo pri $f(x) = x^{-1}$). $(z+2)$ je pri $z = 3$ pozitiven, zato je limita $+\infty$ (in ne $-\infty$).

(f) $\lim_{z\to 4}\left[\frac{\sqrt{z}-2}{z-4}\right] = \lim_{z\to 4}\left[\frac{(\sqrt{z}-2)(\sqrt{z}+2)}{(z-4)(\sqrt{z}+2)}\right] = \lim_{z\to 4}\left[\frac{(z-4)}{(z-4)(\sqrt{z}+2)}\right] =$
$= \lim_{z\to 4}\left[\frac{1}{\sqrt{z}+2}\right] = \frac{1}{4}$

(g) $\lim_{x\to 0}\left[\frac{x}{3-\sqrt{9-x}}\right] = \lim_{x\to 0}\left[\frac{x(3+\sqrt{9-x})}{(3-\sqrt{9-x})(3+\sqrt{9-x})}\right] = \lim_{x\to 0}\left[\frac{x(3+\sqrt{9-x})}{x}\right] =$
$= \lim_{x\to 0}[3+\sqrt{9-x}] = 6$

(h) $\lim_{x\to\infty}\left[\sqrt{x^2+4x}-x\right] = \lim_{x\to\infty}\left[\frac{(\sqrt{x^2+4x}-x)(\sqrt{x^2+4x}+x)}{\sqrt{x^2+4x}+x}\right] =$
$= \lim_{x\to\infty}\left[\frac{4}{\sqrt{1+4/x}+1}\right] = \frac{4}{\sqrt{1+0}+1} = 2$

(i) $\lim_{x\to\infty}\left[x-\sqrt{x^2-7x+2}\right] = \lim_{x\to\infty}\left[\frac{(x-\sqrt{x^2-7x+2})(x+\sqrt{x^2-7x+2})}{x+\sqrt{x^2-7x+2}}\right] =$
$= \lim_{x\to\infty}\left[\frac{x^2-(x^2-7x+2)}{x+\sqrt{x^2+7x+2}}\right] = \lim_{x\to\infty}\left[\frac{7-2/x}{1+\sqrt{1-7/x+2/x^2}}\right] =$
$= \frac{7-0}{1+\sqrt{1-0+0}} = \frac{7}{2}$

(j) $\lim_{x\to 8}\frac{\sqrt[3]{x}-2}{x-8} = \lim_{x\to 8}\frac{\sqrt[3]{x}-2}{(\sqrt[3]{x}-2)((\sqrt[3]{x})^2+2\sqrt[3]{x}+4)} = \lim_{x\to 8}\frac{1}{(\sqrt[3]{x})^2+2\sqrt[3]{x}+4} =$
$= \frac{1}{4+2\cdot 2+4} = \frac{1}{12}$

(k) $\lim_{x\to 0} x^{-1}$. Ta limita ne obstaja, saj ima pol liho stopnjo in velja $\lim_{x\to 0^+} x^{-1} \neq \lim_{x\to 0^-} x^{-1}$

(l) $\lim_{h\to 0}\frac{1/(x+h)-1/x}{h} = \lim_{h\to 0}\frac{x-(x+h)}{hx(x+h)} = -\frac{1}{x^2}$

(m) $\lim_{x\to -\infty}\frac{3x^2-x}{-7x+2} = \lim_{x\to -\infty}\left[-\frac{3}{7}x\right] = +\infty$

2. Limite v fiziki

Naloga 2.1 Gostota ozračja je:

$$\rho = \frac{pM}{RT} = 1{,}124\,\mathrm{kg/m^3}$$

Preoblikujmo enačbo v obliko:

$$v(t) = v_0 \tanh\left(\frac{t}{\tau}\right),$$

kjer

$$v_0 = \sqrt{\frac{C\rho A}{2mg}} = 22{,}6\,\mathrm{m/s}$$

$$\tau = \frac{v_0}{g} = 2{,}3\,\mathrm{s}.$$

(a) Po dolgem času pada žoga s hitrostjo:

$$\lim_{t\to\infty} v_0 \tanh\left(\frac{t}{\tau}\right) = v_0 = 22{,}6\,\mathrm{m/s}.$$

(b)

$$0{,}90 = \frac{v(t)}{v(\infty)} = \tanh\left(\frac{t}{\tau}\right)$$

$$t = \tau \operatorname{artanh} 0{,}90 = 3{,}4\,\mathrm{s}.$$

Naloga 2.2 (a) Po dolgem času je eksponentni člen zanemarljiv in napetost na kondenzatorji je preprosto $V_0 = 9\,\mathrm{V}$.

(b) Vsota napetosti na kondenzatorju in uporniku je V_0. Napetost na uporniku je torej:

$$V_R(t) = V_0 - V(t) = V_0 e^{-t/\tau}.$$

Čas izračunamo tako:

$$2 = \frac{V(t)}{V_R(t)} = \frac{1 - e^{-t/\tau}}{e^{-t/\tau}} = e^{t/\tau} - 1$$

$$t = \tau \ln 3 = 0{,}55\,\mathrm{ms}$$

3. Asimptotsko obnašanje

Naloga 3.1 ⋆ Očitno je potencial zaradi cilindrične simetrije neodvisen od koordinate ϕ. Izračunajmo sedaj potencial v dani točki zaradi naboja $+q$:

$$V_+ = \frac{1}{4\pi\epsilon_0}\frac{q}{r_+} = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{r^2 + a^2/4 - ra\cos\theta}}$$

Na podoben način izračunamo:

$$V_- = \frac{1}{4\pi\epsilon_0}\frac{-q}{r_-} = \frac{1}{4\pi\epsilon_0}\frac{-q}{\sqrt{r^2 + a^2/4 + ra\cos\theta}}$$

Velja torej:

$$V = V(r,\theta) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2 + a^2/4 - ra\cos\theta}} - \frac{1}{\sqrt{r^2 + a^2/4 + ra\cos\theta}}\right)$$

Sedaj predpostavimo, da $a/r \ll 1$.

$$\begin{aligned} V = V(r,\theta) &= \frac{q}{4\pi\epsilon_0 r}\left[\left(1 - \frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right)^{-1/2} - \left(1 + \frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right)^{-1/2}\right] \approx \\ &\approx \frac{q}{4\pi\epsilon_0 r}\left[\left(1 - \frac{a}{r}\cos\theta\right)^{-1/2} - \left(1 + \frac{a}{r}\cos\theta\right)^{-1/2}\right] \approx \\ &\approx \frac{q}{4\pi\epsilon_0 r}\left[1 - \left(-\frac{1}{2}\right)\frac{a}{r}\cos\theta - 1 - \left(-\frac{1}{2}\right)\frac{a}{r}\cos\theta\right] = \\ &= \frac{qa}{4\pi\epsilon_0 r^2} = \frac{p}{4\pi\epsilon_0 r^2}, \end{aligned}$$

kjer smo vpeljali *električni dipolni moment* $p = qa$. Pri električnem monopolu (torej naboju) potencial pada z r^{-1}, pri električnem dipolu pa z r^{-2}. Kompaktno lahko zgornji rezultat zapišemo takole:

$$V(\mathbf{r}) = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3}$$

Naloga 3.2 ⋆ **Wien:** V tem primeru je $h\nu/kT \gg 1$, torej je v izrazu $\exp\left(\frac{h\nu}{kT}\right) - 1$ zadnja enica zanemarljiva. Imamo torej:

$$B_\nu(\nu) \approx \frac{2h\nu^3}{c^2}e^{-h\nu/kT}.$$

Rayleigh-Jeans: Sedaj $h\nu/kT \ll 1$, torej lahko približamo eksponentni člen s potenčno vrsto. En člen ($e^x \approx 1$) je premalo, saj bi v tem primeru dobili v imenovalcu 0. Zato vzamemo dva člena, torej $e^x = 1 + x$, s čimer dobimo:

$$B_\nu(\nu) \approx \frac{2h\nu^3}{c^2}\frac{1}{1 + \frac{h\nu}{kT} - 1} = \frac{2kT\nu^2}{c^2}$$